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Endomorphisms of undirected modifications of directed graphs

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1. Introduction and the main theorems

Let a directed graph $H=(V, A)$ without loops be given, i.e., V is an arbitrary set, $A \subseteq V \times V$ and never $(a, a) \in A$. There are two standard ways how to create an undirected graph from it, namely to create its reflection (its upper modification)

$$\mathscr{R}H=(V, E_{\mathscr{R}}) \quad \text{with } \{v, w\} \in E_{\mathscr{R}} \text{ iff } (v, w) \in A \cup A^{-1}$$

and its coreflection (= lower modification)

$$\mathscr{C}H=(V, H_{\mathscr{C}}) \quad \text{with } \{v, w\} \in E_{\mathscr{C}} \text{ iff } (v, w) \in A \cap A^{-1}.$$

If τ is an automorphism (or an endomorphism) of H, then it is also an automorphism (or an endomorphism) of $\mathscr{R}H$ and of $\mathscr{C}H$. Hence the automorphism group $\operatorname{Aut} H$ is a subgroup of $\operatorname{Aut} \mathscr{R}H$ and of $\operatorname{Aut} \mathscr{C}H$. Is there any other relation between the groups $\operatorname{Aut} H$, $\operatorname{Aut} \mathscr{R}H$ and $\operatorname{Aut} \mathscr{C}H$? More in detail, which spans of groups, see Fig. 1, where m_1, m_2 are monomorphisms, can be realized by directed graphs H (in the sense that there exist isomorphisms φ_0 of G_0 onto $\operatorname{Aut} H$, φ_1 of G_1 onto $\operatorname{Aut} \mathscr{R}H$ and φ_2 of G_2 onto $\operatorname{Aut} \mathscr{C}H$ such that the diagram in Fig. 2—where i_1, i_2 are the inclusion maps—commutes)?

In the present paper, we prove the following.

Theorem 1. *Every span $(*)$ of groups can be realized by a directed graph.*

What we really prove here (in the next section of the present paper) is the following.

Proposition. *Let M be a monoid, let M_1, M_2 be its submonoids and $M_0 = M_1 \cap M_2$. Then there exist a directed graph H and isomorphisms φ_1 of M_1 onto the*

$$s:\quad \begin{tikzcd} & G_1 \\ G_0 \arrow[ur, "m_1"] \arrow[dr, "m_2"'] & \\ & G_2 \end{tikzcd} \quad \Bigg\}\ (*)$$

Fig. 1.

$$\begin{tikzcd} & G_1 \arrow[dd, squiggly] & \\ G_0 \arrow[ur, "m_0"] \arrow[rr, "m_1"] \arrow[dd, squiggly] & & G_2 \arrow[dd, squiggly] \\ & \mathrm{Aut}\,\mathcal{R}H & \\ \mathrm{Aut}\,H \arrow[ur, "i_1"] \arrow[rr, "i_2"] & & \mathrm{Aut}\,\mathcal{C}H \end{tikzcd}$$

Fig. 2.

endomorphism monoid End $\mathcal{R}H$ *and* φ_2 *of* M_2 *onto* End $\mathcal{C}H$ *which coincide on* M_0 *and their joint restriction is an isomorphism of* M_0 *onto* End H.

The theorem is an immediate consequence of the proposition. In fact, by the well-known Schreier theorem (see e.g. [4]), given a span (*) of groups, there is a group G and its subgroups G_1', G_2' isomorphic to G_1, G_2 such that the isomorphisms coincide on $G_1' \cap G_2'$ and the joint restriction is just an isomorphism of $G_1' \cap G_2'$ onto G_0. Then we apply the proposition with $M = G$, $M_1 = G_1'$, $M_2 = G_2'$, $M_0 = G_1' \cap G_2'$.

Unfortunately (or fortunately?), the Schreier theorem is no more valid for monoids, whence the proposition does not imply a characterization of spans of monoids realized by the endomorphism monoids of H, $\mathcal{R}H$ and $\mathcal{C}H$; such a characterization is unknown.

Let us mention that connections between endomorphism monoids (and automorphism groups) of H and of $\mathcal{R}H$ are investigated in [5, 2, 3, 7]. Let us recall some of these results and present comments on them.

(a) By [5], for every pair of monoids $M_0 \subseteq M_1$ (the symbol $\subseteq$ denotes that M_0 is a submonoid of M_1) there exist a directed graph H and an isomorphism of M_1 onto End $\mathcal{R}H$ sending M_0 onto End H. The 'dual result', i.e., replacing $\mathcal{R}H$ in the above statement by $\mathcal{C}H$, is also valid. Both the statements are special cases of our proposition: the result of [5] is obtained by the choice $M_2 = M_0 \subseteq M_1 = M$, the 'dual result' by the exchange of the rôle of M_1 and M_2.

(b) By [5], for every pair of groups $G_0 \subseteq G_1$ (i.e., G_0 is a subgroup of G_1) there exist a directed graph H with $\mathcal{C}H$ being a discrete graph (i.e., no 2-cycle is in H) and an isomorphism of G_1 onto Aut $\mathcal{R}H$ sending G_0 onto Aut H.

Let us formulate the 'dual result': for every pair of groups $G_0 \subseteq G_1$ there exist a directed graph H with $\mathcal{R}H$ being a complete graph and an isomorphism of G_1 onto Aut $\mathcal{C}H$ sending G_0 onto Aut H. This dual result is obtained immediately from the previous one: we find a graph H satisfying the original statement of [5] and take its complement $-H$.

(c) In [3], a stronger result than that of (b) is proved: given a monoid M_1 and its *subgroup* M_0, there exist a directed graph H with $\mathcal{C}H$ discrete and an isomorphism of M_1 onto End $\mathcal{R}H$ sending M_0 onto End H. Also, examples are given in [2, 3] that M_0 cannot be an arbitrary *submonoid* of M_1. Which monoid pairs $M_0 \subseteq M_1$ can be realized in this way? The theorem below gives an answer.

Theorem 2. *A monoid pair $M_0 \subseteq M_1$ can be realized by a directed graph (in the sense that there exists a directed graph H with $\mathscr{C}H$ discrete and an isomorphism of M_1 onto* End $\mathscr{R}H$ *sending M_0 onto* End H) *iff M_0 is a submonoid of M_1 satisfying the following implication*:

$$m_0 \in M_0,\ m_1 \in M_1 \backslash M_0 \Rightarrow m_0 \cdot m_1 \in M_1 \backslash M_0. \tag{**}$$

The proof of Theorem 2 is also presented in Section 2. Let us notice that the 'dual problem' (i.e., a characterization of the monoid pairs $M_0 \subseteq M_1$ realizable by a directed graph H with $\mathscr{R}H$ being a complete graph and M_1 isomorphic to End $\mathscr{C}H$ by an isomorphism sending M_0 onto End H) is unsolved. In this connection, let us ask here the following question which also seems to be unsolved: for which monoid pairs (M_0, M_1) there exists a graph H such that $M_0 \simeq \text{End}\, H$ and $M_1 \simeq \text{End}\, -H$ (where $-H$ is the graph complementary to H)?

2. The proofs

Proof of the proposition. (1) Let a monoid M and its submonoids M_1, M_2, $M_0 = M_1 \cap M_2$ be given. By [7, Lemma II.4], there are

(i) a quintuple $Q = (X, R_0, R_1, R_2, R)$ such that (X, R) is a connected directed graph without loops, $R_i \subseteq R$ for $i = 1, 2$, $R_0 = R_1 \cap R_2$ and

(ii) an isomorphism φ of M onto End(X, R) sending M_i onto End(X, R_i, R) (= End$(X, R_i) \cap$ End(X, R)) for $i = 1, 2$, hence M_0 onto End(X, R_0, R_1, R_2, R).

We construct the required directed graph $H = (V, A)$ by means of the 'arrow construction', i.e., we replace arrows in R by suitable directed graphs. However, we choose distinct graphs K_0, K_1, K_2, K to replace arrows in R_0, $R_1 \backslash R_0$, $R_2 \backslash R_0$, $R \backslash (R_1 \cup R_2)$.

(2) First, let us construct the graphs K, K_i, $i = 0, 1, 2$. We start from an auxiliary graph L consisting of one 3-cycle and one 5-cycle having one arrow in common (as indicated in Fig. 3). Let the arrows in the 5-cycle disjoint with the 3-cycle be named (consecutively) p, q, t, s. Let (Y, S) be an undirected graph consisting of three 7-cycles having one or two or three edges in common, see Fig. 4. This graph is described with all details in [6, p. 68], hence we present here only its picture (with the same names of vertices as in [6, p. 68]; let us mention

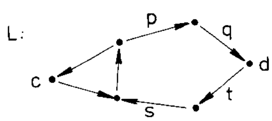

Fig. 3.

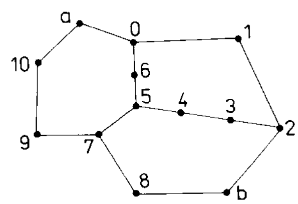

Fig. 4.

explicitly that the vertex named 5 is the unique vertex which belongs to each of the three 7-cycles).

Finally, let us replace each arrow in L by a copy of (Y, S), taking disjoint copies of (Y, S) for distinct arrows and then identifying vertex a and b with the initial and the terminal vertex of the replaced arrow (this arrow construction is described with all details in [6, pp. 105–107]). By [6], let us denote the resulting graph by $L*(Y, S)$. Let us denote by 5_p (or 5_q or ... or 5_s) the vertex 5 of the copy of (Y, S) replacing the arrow p (or q or ... or s). Denote $(W, T) = L*(Y, S)$ and let us investigate it as a directed symmetric graph. We put

$$K = (W, T \cup \{(5_p, 5_t)\}),$$
$$K_1 = (W, T \cup \{(5_p, 5_t), (5_q, 5_s)\}),$$
$$K_2 = (W, T \cup \{(5_p, 5_t), (5_t, 5_p)\}),$$
$$K_0 = (W, T \cup \{(5_p, 5_t), (5_t, 5_p), (5_q, 5_s)\}).$$

Moreover, put $F = T \cup \{(5_p, 5_t), (5_t, 5_p), (5_q, 5_s), (5_s, 5_q)\}$. One can see easily that

$$\mathscr{C}K_0 = \mathscr{C}K_2 = K_2; \qquad \mathscr{R}K = \mathscr{R}K_2 = K_2;$$
$$\mathscr{C}K_1 = \mathscr{C}K = (W, T); \qquad \mathscr{R}K_0 = \mathscr{R}K_1 = (W, F).$$

Let us denote by c and d the vertices of K, K_1, K_2, K_0 as indicated on the picture of the auxiliary graph L (this makes sense because the set of vertices of L can be regarded as a subset of the set of vertices $(W, T) = L*(Y, S)$, namely the set of vertices in which the copies of (Y, S) are 'glued together'). Finally, we denote by P_r a copy of

–K_0 whenever $r \in R_0$,

–K_i whenever $r \in R_i \setminus R_0$, $i = 1, 2$,

–K whenever $r \in R \setminus (R_1 \cup R_2)$

(such that if $r \neq r'$, then P_r and $P_{r'}$ are disjoint) and replace each arrow r in R by P_r identifying the vertices c and d with the initial and the terminal vertices of r. The obtained directed graph $H = (V, A)$ will be shown to have all the required properties.

(3) One can see easily that $\mathrm{End}(X, R)$ is isomorphic to $\mathrm{End}((X, R)*L)$ and the isomorphism sends $\mathrm{End}(X, R_i, R)$ onto $\mathrm{End}((X, R_i, R)*L)$, $i = 1, 2$ and analogously for $\mathrm{End}(X, R_0, R_1, R_2, R)$ (this follows from the fact that any endomorphism of $(X, R)*L$ has to send each 3-cycle onto a 3-cycle and each 5-cycle onto a 5-cycle, hence a copy of L identically onto a copy of itself, so it determines uniquely an endomorphism of (X, R)). Analogously as in [6, p. 70], one can see that $\mathrm{End}((X, R)*L)*(Y, S))$ is isomorphic to $\mathrm{End}((X, R)*L)$. The proof is based on the fact that $((X, R)*L)*(Y, S)$ has no shorter odd cycles than 7-cycles, hence any endomorphism sends any 7-cycle onto a 7-cycle; a small reasoning shows that it must send each copy of (Y, S) identically onto a copy of itself so that it determines uniquely an endomorphism of $(X, R)*L$. But

$((X, R)*L)*(Y, S) = (X, R)*(W, T)$. Since the adding of the arrows $(5_p, 5_t), \ldots, (5_s, 5_q)$ does not create any new short odd cycle, the same arguments can be used to show that any endomorphism of H necessarily sends a copy of
–K_0 identically onto a copy of itself,
–K_i identically onto a copy of K_0 or a copy of itself, $i = 1, 2$,
–K identically onto a copy of K_0 or K_1 or K_2 or itself,
any endomorphism of $\mathcal{R}H$ necessarily sends a copy of
–(W, F) identically onto a copy of itself,
–K_2 identically onto a copy of (W, F) or itself,
and, finally, any endomorphism of $\mathcal{C}H$ necessarily sends a copy of
–K_2 identically onto a copy of itself,
–(W, T) identically onto a copy of K_2 or itself.
Hence $\operatorname{End} H \simeq \operatorname{End}(X, R_0, R_1, R_2, R)$ and the isomorphism can be extended to the isomorphism $\operatorname{End} \mathcal{R}H \simeq \operatorname{End}(X, R_1, R)$ and $\operatorname{End} \mathcal{C}H \simeq \operatorname{End}(X, R_2, R)$, which concludes the proof. □

Proof of Theorem 2. (1) First, let us show that the condition of Theorem 2 is necessary. Thus, let $H = (V, A)$ be a directed graph without loops such that $\mathcal{C}H$ is discrete, i.e., $A \cap A^{-1} = \emptyset$, and $M_1 = \operatorname{End} \mathcal{R}H$, $M_0 = \operatorname{End} H$. Let $m_0 \in M_0$, $m_1 \in M_1 \setminus M_0$ be given. Since $m_1 \notin M_0$, there exists $(v, w) \in A$ such that $(m_1(v), m_1(w)) \notin A$. But $m_1 \in M_1$, so that necessarily $\{m_1(v), m_1(w)\} \in E_{\mathcal{R}}$, consequently $(m_1(w), m_1(v)) \in A$. Then $m_0 \in M_0$ implies that $(m_0(m_1(w)), m_0(m_1(v))) \in A$. Since $\mathcal{C}H = \emptyset$, necessarily $(m_0(m_1(v)), m_0(m_1(w))) \notin A$, hence $m_0 \cdot m_1 \notin M_0$.

(2) Now, we prove that the condition of Theorem 2 is sufficient. Thus, let $M_0 \subseteq M_1$ satisfying (**) be given. First, we find a directed graph without loops (Z, Q) and an isomorphism φ of M_1 onto (Z, Q) with the following properties:

(i) Z contains two disjoint copies of M_1, say $[M_1]^1$ and $[M_1]^2$ such that no vertex in $[M_1]^1$ is joined by an arrow with any vertex of $[M_1]^2$ and vice versa;

(ii) for each $m \in M_1$, $\varphi(m)$ acts as a left translation on the both copies of M_1, i.e., for every $n \in M_1$,

$$(\varphi(m))([n]^1) = [m \cdot n]^1, \qquad (\varphi(m))([n]^2) = [m \cdot n]^2,$$

where by $[n]^1$, $[n]^2$ we denote the element $n \in M_1$ in the copy $[M_1]^1$ and $[M_1]^2$.

A graph (Z, Q) with these properties really does exist. (In fact, we can take a graph $(\bar{Z}, \bar{Q})$ containing one such copy of M_1 (see e.g. [6, p. 76]); moreover, we may suppose that $(\bar{Z}, \bar{Q})$ is acyclic (see e.g. [6, p. 108]); then we take two disjoint copies of $(\bar{Z}, \bar{Q})$, glue the graph L_5 at c on each vertex of the first copy of $(\bar{Z}, \bar{Q})$ and the graph L_7 at c' on each vertex of the second copy of $(\bar{Z}, \bar{Q})$ (where L_n is the graph consisting of a 3-cyclic and an n-cycle having one arrow in common; L_5 is just the auxiliary graph L from the previous proof; the vertices c, d are situated in L_5 as indicated on the picture of L, the vertices c', d' are situated in L_7

analogously) and finally join the vertex d of each copy of L_5 with the vertex d' of the corresponding copy (i.e., glued on the same vertex of the second copy of (Z, Q)) of L_7).

(3) Let (W, T) be the graph constructed in the previous proof, i.e., $(W, T) = L * (Y, S)$. Then every edge lies in a 7-cycle, there are no shorter odd cycles in (W, T) and the distance of c and d in (W, T) is equal to 12. Hence, if we use the arrow construction again and create the graph

$$(V, D) = (Z, Q) * (W, T),$$

then still (V, D) contains the two copies $[M_1]^1$ and $[M_1]^2$ and M_1 is isomorphic to $\mathrm{End}(V, D)$ by an isomorphism ψ satisfying (ii) (where we replace the letter φ by ψ), but (V, D) is an undirected graph and, since the distance of any two distinct elements of $[M_1]^1 \cup [M_1]^2$ is at least 12 in (V, D), while the endomorphisms of (V, D) are determined by the mutual position of 7-cycles, no added edge joining a vertex of $[M_1]^1$ and of $[M_1]^2$ can destroy the fact that any endomorphism of the newly obtained graph (i.e., with these edges added) has to send any copy of (W, T) identically onto a copy of itself, i.e., the endomorphism is necessarily in $\mathrm{End}(V, D)$. Hence, if we put

$$B = D \cup \{\{[m]^1, [m]^2\} \mid m \in M_1\},$$

then ψ is also an isomorphism of M_1 onto $\mathrm{End}(V, B)$. We are going to construct the graph H with the required properties such that $\mathcal{R}H$ will be precisely (V, B). We proceed as follows: first, we choose one (arbitrary but fixed) orientation of the graph (Y, S) (i.e., for every $\{y_1, y_2\} \in S$ we choose either (y_1, y_2) or (y_2, y_1) but not both), denote by $(Y, \tilde{S})$ the obtained directed graph. Clearly $\mathscr{C}(Y, \tilde{S})$ is discrete. Then denote $(W, \tilde{T}) = L * (Y, \tilde{S})$ and $(V, \tilde{D}) = (Z, Q) * (W, \tilde{T})$. Finally put $H = (V, A)$, where $A = \tilde{D} \cup C$ and

$$C = \{([m]^1, [m]^2) \mid m \in M_0\} \cup \{([m]^2, [m]^1) \mid m \in M_1 \setminus M_0\}.$$

(4) Clearly, $\mathscr{C}H$ is discrete and $\mathcal{R}H = (V, B)$. It remains to show that the above isomorphism ψ of M_1 onto $\mathrm{End}(V, B)$ sends M_0 onto $\mathrm{End}(V, A)$. However, for any $n \in M_1$, $\psi(n)$ acts on $[M_1]^1$ and on $[M_1]^2$ as the left translation. This implies that $\psi(n)$ preserves C iff n is in M_0. We show it more in detail.

(α) If $n \in M_0$, then $\psi(n) \in \mathrm{End}(V, A)$: in fact, if $([m]^1, [m]^2) \in C$, then $m \in M_0$, hence $n \cdot m \in M_0$ so that

$$((\psi(n))([m]^1), (\psi(n))([m]^2)) = ([n \cdot m]^1, [n \cdot m]^2) \in C;$$

if $([m]^2, [m]^1) \in C$, then $m \in M_1 \setminus M_0$, hence $n \cdot m \in M_1 \setminus M_0$ by (**), so that the $\psi(n)$-image $([n \cdot m]^2, [n \cdot m]^1)$ is in C again.

(β) If $n \in M_1 \setminus M_0$, then $\psi(n) \notin \mathrm{End}(V, A)$: in fact, $([1]^1, [1]^2) \in C$ while $([n]^1, [n]^2) \notin C$. □

Concluding remarks

In [5, 3], more general results are proved than those quoted here in (a), (b), (c) in Section 1. They express the fact that the constructions are 'uniform'. The construction in the proof of Theorem 2 is also 'uniform' so that we have proved the following result: for every monoid M_1 there exists an undirected graph G with $\operatorname{End} G \simeq M_1$ such that for every submonoid M_0 of M_1 satisfying (**) there exists a directed graph H with $\mathscr{R}H = G$, $\mathscr{C}H$ discrete such that $\operatorname{End} H \simeq M_0$; moreover, the poset of all the submonoids of M_1 satisfying (**) (ordered by the inclusion) is isomorphic to the poset of all monoids $\operatorname{End} H$, where $\mathscr{R}H = G$, $\mathscr{C}H$ is discrete (but there are distinct H, H' with $\operatorname{End} H = \operatorname{End} H'$ of course). Both the Proposition and Theorem 2 could also be generalized from monoids to small categories.

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